Multivariable Calculus with Maxima

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The following is a short guide to multivariable calculus with Maxima. It loosely follows the treatment of Stewart's *Calculus*, Seventh Edition. Refer there for definitions, theorems, proofs, explanations, and exercises. The simple goal of this guide is to demonstrate how to use Maxima to solve problems in that vein.

This was originally written for the students in my third semester Calculus class, but once it grew past twenty pages I thought it might be of interest to a wider audience. Here it is. I am releasing this as a FREE document, and other people are free to build on this to make it better. The source for this document is located at

http://people.ysu.edu/~gkerns/maxima/

It was inspired by *Maxima by Example* by Edwin Woollett, *A Maxima Guide for Calculus Students* by Moses Glasner, and *Tutorial on Maxima* by unknown. I also received help from the Maxima mailing list archives and volunteer responses to my questions. Thanks to all of those individuals.

Contents

1 Getting Maxima

There are many ways to get Maxima, and the choices are governed somewhat by the user's operating system (although Maxima proper is platform independent). The home page for Maxima (which has links to download from SourceForge) is

http://maxima.sourceforge.net

This document was written with Emaxima on GNU-Emacs. Emacs is a powerful text editor, and Emaxima uses Emacs to integrate Maxima input/output with LaTeX code to produce documents that look like they could have come from a textbook. That's why the mathematical expressions below are so pretty. See the following link to learn more about Emaxima.

http://www.emacswiki.org/emacs/MaximaMode

If you do not want to write a paper but just want to work with Maxima interactively then another option is imaxima, which also works with Emacs and LaTeX (and Ghostscript) to typeset Maxima input/output professionally. To learn more about imaxima see the following.

http://sites.google.com/site/imaximaimath/

Another option that exists is TeXmacs, but I do not have much experience with it. From what I can gather it has a good reputation.

1.1 How to Install imaxima for Microsoft Windows®

The following instructions are to set up imaxima on a computer with a Microsoft Windows® operating system (the majority of users). If you are running Mac-OS then instead go to

http://sites.google.com/site/imaximaimath/download-and-install/easy-install-on-mac-os-x

and if you are running Linux (like me) you can go to

http://sites.google.com/site/imaximaimath/download-and-install/easy-install-on-linux.

Please note that these instructions are NOT needed to install plain-old Maxima. It is already installed in Cushwa 1062, and you can install it at home with the 'Maxima' instructions in the next subsection.

1.1.1 Install the software

In order to take advantage of the full power of imaxima you need several things.

MiKTEX 2.7 (pronounced "MICK - teck") is an up-to-date implementation of TEX and related programs for Windows®. Its official web site is `www.miktex.org`. You can go to `http://www.miktex.org/2.7/Setup.aspx` and click the Download button of the `Basic MiKTeX 2.7 Installer` on that page. You may save the `.exe` file anywhere, such as the Desktop. Start the installation with a double-click. You need to install it in the default place.

GPL Ghostscript 8.63 The main page for Ghostscript is `http://www.cs.wisc.edu/~ghost/`. To download the latest Ghostscript 8.64 compiled for Windows®, you go to

`http://pages.cs.wisc.edu/~ghost/doc/GPL/gpl864.htm`.

In the middle of the page there there is a link to the file you need. Choose `gs864w32.exe` (or the latest release; if you have a 64-bit system then you will need `gs864w64.exe`, and if you do not know what I am talking about then you probably do not need the 64-bit version). You can double click the downloaded `gs864w32.exe` file to start the installation. You need to install it in the default place.

Maxima Go to `http://sourceforge.net/projects/maxima/files/`, and scroll down to the `Maxima-Windows` section. Choose `maxima-5.19.2.exe` (or the latest release) for the download of a Windows® pre-compiled binary installer. Double click the downloaded `maxima-5.19.2.exe` file to start installation. You need to install it in the default place.

Emacs is a very powerful text editor. The very official precompiled release can be obained from `http://ftp.gnu.org/pub/gnu/emacs/windows/`. However, the distributed precompiled binary does NOT support image features, hence it is not good enough for imaxima, by default. Thanks to Vincent Goulet, however, you can obtain a precompiled binary installer which has everything you need to use imaxima. Go to the web page `http://vgoulet.act.ulaval.ca/en/ressources/emacs/windows` and choose `emacs-23.1-modified-3.exe` for download. Just follow the instructions and install it in the default place.

1.1.2 Configure your system

Once you have installed all of the above, go to the Start Menu under GNU Emacs and click Update Site Configuration. In the file that opens add the following single line (and it has to be *exactly* right) to the top of the file and click the Save button. Note that there shouldn't be any line break; it is one, big, long, line.

```
(load "C:/Program Files/Maxima-5.19.2/share/maxima/5.19.2/emacs
   /setup-imaxima-imath.el")
```

If you have installed a different version of Maxima then the `5.19.2` appearing twice in the above line should be replaced with the version you have downloaded.

Then quit Emacs and restart. When it opens, type:

```
M-x imaxima
```

The symbol `M` means the Alt key (which is also known as the Meta key). Thus, the command `M-x imaxima` means

1. Hold down the Alt key and press `x`.
2. Let go, then type `imaxima`, then press Enter.

If this is your first time using imaxima, MikTEX will probably ask you to install the `mh` package from CTAN. Please proceed with Yes for installation. This may take some time, so be patient.

Then, you will see the initial screen of Maxima. Enjoy!!

2 Three Dimensional Geometry

2.1 Vectors and Linear Algebra

We set up vectors in a way which is slightly different from how we do it in Maple.

```
(%i1) a: [1,2,3];
(%o1)
```

$$[1, 2, 3]$$

```
(%i2) b: [2,-1,4];
(%o2)
```

$$[2, -1, 4]$$

We can do the standard addition and scalar multiplication of vectors.

```
(%i3) 3 * a;
(%o3)
```

$$[3, 6, 9]$$

```
(%i4) a + b;
(%o4)
```

$$[3, 1, 7]$$

The dot product operator is a simple period "." between the vectors.

```
(%i5) a . b;
```

(%o5)

$$12$$

We can check our answer to make sure it is right.

```
(%i6) 1 * 2 + 2 * (-1) + 3 * 4;
(%o6)
```

$$12$$

To do cross products we must load a special package, the `vect` package, which is included with Maxima.

```
(%i7) load(vect);
(%o7)
```

`/usr/local/share/maxima/5.19.2/share/vector/vect.mac`

The symbol for the cross product is the tilde "~" up on the left corner of the keyboard (you need to do Shift to get it).

```
(%i8) a ~ b;
(%o8)
```

$$([1, 2, 3], [2, -1, 4])$$

```
(%i9) express(%);
(%o9)
```

$$[11, 2, -5]$$

The cross product did not look like anything at the beginning; we had to `express` the cross product to get something recognizable.

The norm (length) of a vector is the square root of the dot product of the vector with itself.

```
(%i10) sqrt(a . a);
(%o10)
```

$$\sqrt{14}$$

Putting what we have learned all together we may do vector projections and scalar triple products (we will need another vector `c` for the STP to make sense).

```
(%i11) (a . b)/(a . a) * a;
(%o11)
```

$$\left[\frac{6}{7}, \frac{12}{7}, \frac{18}{7}\right]$$

```
(%i12) c: [-5, 2, 9];
(%o12)
```

$$[-5, 2, 9]$$

```
(%i13) a . (b ~ c);
(%o13)
```

$$[1, 2, 3] \cdot \quad ([-2, 1, -4], [5, -2, -9])$$

```
(%i14) express(%);
(%o14)
```

$$-96$$

Again, we needed to `express` the cross product to get anything useful.

2.2 Lines, Planes, and Quadric Surfaces

We can plot planes with `implicitplot` from the `draw` package.

First let us define the plane with equation

$$3x + 4y + 5z = 0.$$

We store the equation of the plane in the variable `plane1`.

```
(%i1) plane1: 3*x + 4*y + 5*z = 0;
(%o1)
```

$$5\,z + 4\,y + 3\,x = 0$$

```
(%i2) load(draw);
(%o2)
```

`/usr/local/share/maxima/5.19.2/share/draw/draw.lisp`

```
(%i3) draw3d(enhanced3d = true, implicit(plane1, x,-4,4, y,-4,4, z, -6,6));
(%o3)
```

$$[\mathrm{gr3d}\,(\mathrm{implicit})]$$

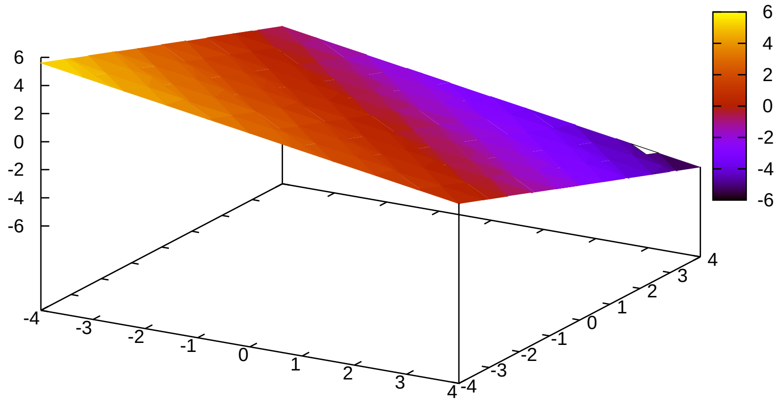

Figure 1: A plot of a plane

Let us next plot an ellipsoid with equation

$$\frac{x^2}{3} + y^2 + z^2 = 3.$$

We plot it just like we plot the plane.

```
(%i4)  ellips1: x^2/3 + y^2 + z^2 = 3;
(%o4)
```

$$z^2 + y^2 + \frac{x^2}{3} = 3$$

```
(%i5)  draw3d(enhanced3d = true, implicit(ellips1, x,-3,3, y,-2,2, z, -2,2));
(%o5)
```

[gr3d (implicit)]

We can also use Maxima to help us find an equation of a plane based on defining vectors. For instance, let's find and plot the plane determined by the points $A(1, 1, 1)$, $B(1, 2, 3)$, and $C(0, 0, 0)$.

First we define the position vectors for the three defining points.

```
(%i6)  a: [1, 1, 1];
(%o6)
```

$[1, 1, 1]$

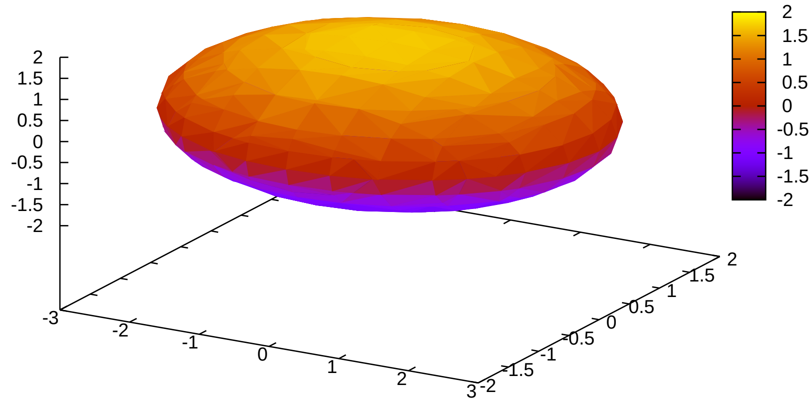

Figure 2: A plot of an ellipsoid

```
(%i7)  b: [1, 2, 3];
(%o7)
```

$$[1, 2, 3]$$

```
(%i8)  c: [0, 0, 0];
(%o8)
```

$$[0, 0, 0]$$

Next, we find the vectors from A to B, and A to C.

```
(%i9)  ab: b - a;
(%o9)
```

$$[0, 1, 2]$$

```
(%i10)  ac: c - a;
(%o10)
```

$$[-1, -1, -1]$$

Then, we find the normal vector to the plane. (Recall that we need the `vect` package to do cross products.)

```
(%i11)  load(vect);
```

(%o11)

/usr/local/share/maxima/5.19.2/share/vector/vect.mac

(%i12) `n: express(ab ~ ac);`
(%o12)

$$[1, -2, 1]$$

Finally, we set up the defining equation of the plane, which takes the form

$$\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r_0}.$$

(%i13) `r: [x, y, z];`
(%o13)

$$[x, y, z]$$

(%i14) `r0: a;`
(%o14)

$$[1, 1, 1]$$

(%i15) `plane: n . r = n . r0;`
(%o15)

$$z - 2\,y + x = 0$$

The only remaining task is to make the plot. See Figure BLANK.

(%i16) `draw3d(enhanced3d = true,implicit(plane,x,-4,4,y,-4,4,z,-4,4));`
(%o16)

$$[\mathrm{gr3d}\,(\mathrm{implicit})]$$

We can do more exotic plots, like cones. Let's do a standard cone with equation

$$x^2 + y^2 = z^2$$

(%i17) `cone: x^2 + y^2 = z^2;`
(%o17)

$$y^2 + x^2 = z^2$$

(%i18) `draw3d(enhanced3d = true, implicit(cone, x,-1,1, y,-1,1, z,-0.5,0.5));`

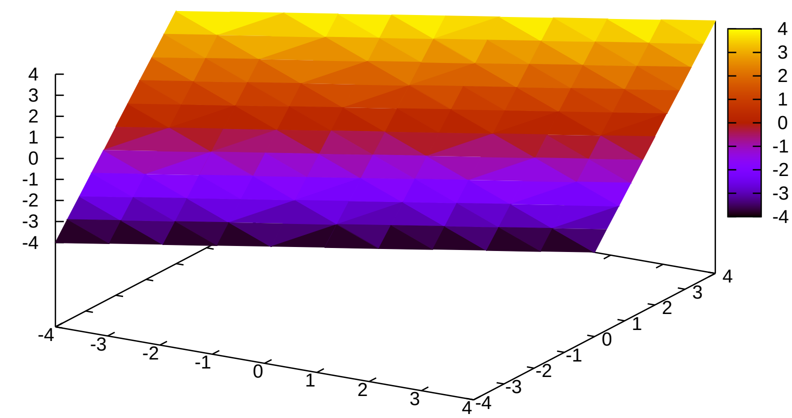

Figure 3: Another plot of a plane

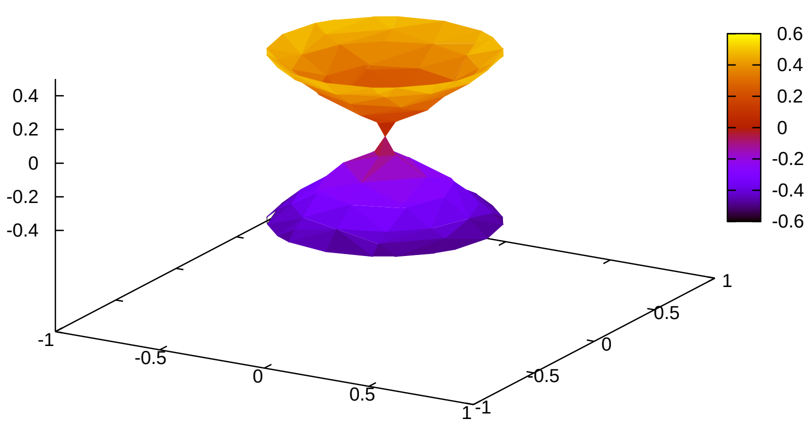

Figure 4: A plot of a cone

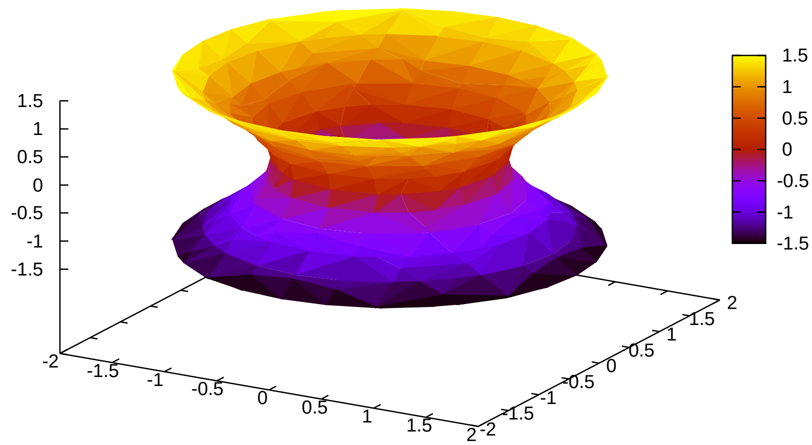

Figure 5: A plot of a hyperboloid

(%o18)

[gr3d (implicit)]

See Figure BLANK. See how the center of the cone looks distorted? It is because we are plotting the cone in rectangular coordinates. We get a much better plot with spherical coordinates. More on that later.

Let's next try a hyperboloid with equation

$$x^2 + y^2 - z^2 = 1$$

See Figure BLANK.

```
(%i19) hyperboloid: x^2 + y^2 - z^2 = 1;
(%o19)
```

$$-z^2 + y^2 + x^2 = 1$$

```
(%i20) draw3d(enhanced3d = true, implicit(hyperboloid, x,-2,2, y,-2,2, z,-1.5,1.5));
(%o20)
```

[gr3d (implicit)]

And we can do a hyperboloid of two sheets, with the equation

$$-x^2 - y^2 + z^2 = 1.$$

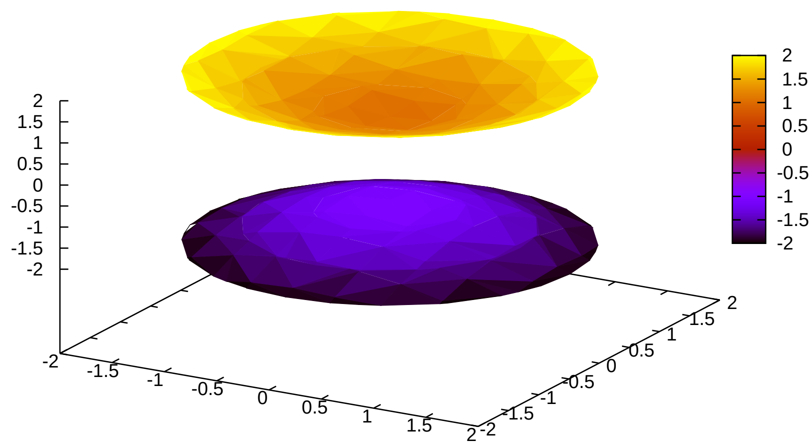

Figure 6: Another plot of a hyperboloid

See Figure BLANK.

```
(%i21) hyprbld2: -x^2 - y^2 + z^2 = 1;
(%o21)
```

$$z^2 - y^2 - x^2 = 1$$

```
(%i22) draw3d(enhanced3d = true, implicit(hyprbld2, x,-2,2, y,-2,2, z, -2,2));
(%o22)
```

$$[\text{gr3d}\,(\text{implicit})]$$

2.3 Vector Valued Functions

We define vector functions in Maxima just like we would any other function; the only difference is that the return value is a vector.

```
(%i1) r(t) := [t, cos(t), sin(t)];
(%o1)
```

$$r\,(t)\ := [t, \cos t, \sin t]$$

Once we have the function we can plug in values of t to evaluate.

```
(%i2) r(1);
```

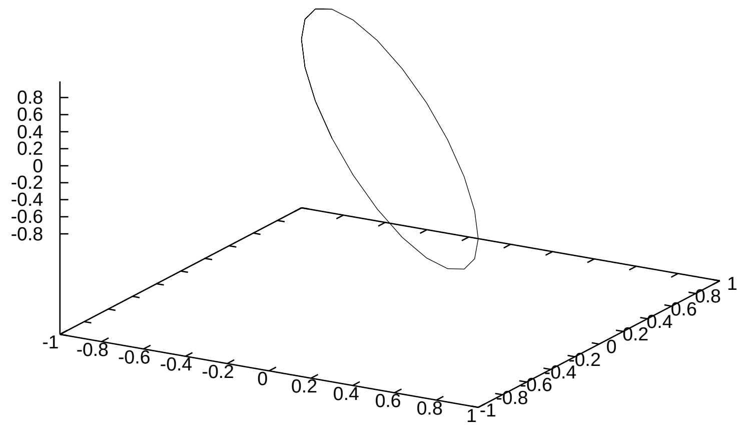

Figure 7: A space curve in 3-d

(%o2)

$$[1, \cos 1, \sin 1]$$

(%i3) `float(%);`
(%o3)

$$[1.0, .5403023058681398, .8414709848078965]$$

Now let's try some plotting. Let's first try a space curve of the vector function. We need the `draw` package to make this happen. Once we have it loaded, we do space curves with the `parametric` function inside of a `draw` command.

(%i4) `load(draw);`
(%o4)

`/usr/local/share/maxima/5.19.2/share/draw/draw.lisp`

(%i5) `draw3d(parametric(cos(t), -cos(t), sin(t), t, -4, 4));`
(%o5)

$$[\mathrm{gr3d}\ (\mathrm{parametric})]$$

See Figure 1. Sometimes the line width of a space curve leaves something to be desired. We can fix the issue with the `line width` argument to the `draw` command. See Figure 2.

Limits of vector functions operate just like limits of everything else. Notice below how to do limits from the right and left.

```
(%i6)  limit(r(t), t, 2);
(%o6)
```

$$[2, \cos 2, \sin 2]$$

```
(%i7)  float(%);
(%o7)
```

$$[2.0, -.4161468365471424, .9092974268256817]$$

```
(%i8)  limit(r(t), t, 2, plus);
(%o8)
```

$$[2, \cos 2, \sin 2]$$

```
(%i9)  limit(r(t), t, 3, minus);
(%o9)
```

$$[3, \cos 3, \sin 3]$$

Derivatives of vector-valued functions are, again, just like their one-valued counterparts.

```
(%i10)  diff(r(t), t);
(%o10)
```

$$[1, -\sin t, \cos t]$$

We need to be careful, here. In both Maple and Maxima the return value of `diff(r(t),t)` is an *expression*, and is NOT a function (although it sure does look like one). Looks are deceptive in this case. Watch what happens if we try to use it like a function.

```
(%i11)  wrong(t) := diff(r(t),t);
(%o11)
```

$$\mathrm{wrong}\,(t) \;:= \mathrm{diff}\,(r\,(t)\,,t)$$

```
(%i12)  wrong(1);
```

** error while printing error message **
~:M: second argument must be a variable; found ~M
#0: wrong(t=1)
– an error. To debug this try debugmode(true);

This is not right. The way to get around this in Maple is to do `D(r(t),t)`, and the way to get around this in Maxima is to do the following.

```
(%i13) define(rp(t), diff(r(t), t));
(%o13)
```

$$\mathrm{rp}\,(t) := [1, -\sin t, \cos t]$$

Now we get what we expect.

```
(%i14) float(rp(1));
(%o14)
```

$$[1.0, -.8414709848078965, .5403023058681398]$$

We may define the unit tangent vector **T** as the normalized derivative of **r**. We do this with the `uvect` function in the `eigen` package.

```
(%i15) load(eigen);
(%o15)
```

`/usr/local/share/maxima/5.19.2/share/matrix/eigen.mac`

```
(%i16) uvect(rp(t));
(%o16)
```

$$\left[\frac{1}{\sqrt{(\sin t)^{2} + (\cos t)^{2} + 1}}, -\frac{\sin t}{\sqrt{(\sin t)^{2} + (\cos t)^{2} + 1}}, \frac{\cos t}{\sqrt{(\sin t)^{2} + (\cos t)^{2} + 1}}\right]$$

```
(%i17) trigsimp(%);
(%o17)
```

$$\left[\frac{1}{\sqrt{2}}, -\frac{\sin t}{\sqrt{2}}, \frac{\cos t}{\sqrt{2}}\right]$$

```
(%i18) define(T(t), %);
(%o18)
```

$$T\,(t) := \left[\frac{1}{\sqrt{2}}, -\frac{\sin t}{\sqrt{2}}, \frac{\cos t}{\sqrt{2}}\right]$$

To get the unit normal vector we need to get the derivative of **T** and normalize.

```
(%i19) define(Tp(t), diff(T(t), t));
(%o19)
```

$$\mathrm{Tp}\,(t) := \left[0, -\frac{\cos t}{\sqrt{2}}, -\frac{\sin t}{\sqrt{2}}\right]$$

(%i20) uvect(Tp(t));

(%o20)

$$\left[0, -\frac{\cos t}{\sqrt{(\sin t)^{2} + (\cos t)^{2}}}, -\frac{\sin t}{\sqrt{(\sin t)^{2} + (\cos t)^{2}}}\right]$$

(%i21) trigsimp(%);

(%o21)

$$[0, -\cos t, -\sin t]$$

(%i22) define(N(t), %);

(%o22)

$$N(t) := [0, -\cos t, -\sin t]$$

Now we find the binormal vector, **B**, by calculating the cross product of **T** and **N**. Remember to do `load(vect)` if you haven't already during the session.

(%i23) load(vect);

(%o23)

`/usr/local/share/maxima/5.19.2/share/vector/vect.mac`

(%i24) express(T(t) ~ N(t));

(%o24)

$$\left[\frac{(\sin t)^{2}}{\sqrt{2}} + \frac{(\cos t)^{2}}{\sqrt{2}}, \frac{\sin t}{\sqrt{2}}, -\frac{\cos t}{\sqrt{2}}\right]$$

(%i25) trigsimp(%);

(%o25)

$$\left[\frac{1}{\sqrt{2}}, \frac{\sin t}{\sqrt{2}}, -\frac{\cos t}{\sqrt{2}}\right]$$

(%i26) define(B(t), %);

(%o26)

$$B(t) := \left[\frac{1}{\sqrt{2}}, \frac{\sin t}{\sqrt{2}}, -\frac{\cos t}{\sqrt{2}}\right]$$

(%i27) float(B(1));

(%o27)

$$[.7071067811865475, .5950098395293859, -.3820514243700898]$$

2.4 Arc Length and Curvature

There is no special Maxima function for the curvature but we can do it with the formulas we learned in class. Recall from the last section that $\mathbf{r}(t) = \langle t,\ \cos\, t,\ \sin\, t\rangle$ and

$$\mathbf{T}(t) \;= \left\langle \frac{1}{\sqrt{2}},\ -\frac{\sin\, t}{\sqrt{2}},\ \frac{\cos\, t}{\sqrt{2}} \right\rangle.$$

with derivative

$$\mathbf{T}'(t) \;= \left\langle 0,\ -\frac{\cos\, t}{\sqrt{2}},\ -\frac{\sin\, t}{\sqrt{2}} \right\rangle.$$

```
(%i1) r(t)  := [t, cos(t), sin(t)];
(%o1)
```

$$r\left(t\right) \;:= [t, \cos t, \sin t]$$

```
(%i2) rp(t) := [1 ,-sin(t), cos(t)];
(%o2)
```

$$\mathrm{rp}\left(t\right) \;:= [1, -\sin t, \cos t]$$

```
(%i3) Tp(t) := [0, -cos(t), sin(t)]/sqrt(2);
(%o3)
```

$$\mathrm{Tp}\left(t\right) \;:= \frac{[0, -\cos t, \sin t]}{\sqrt{2}}$$

```
(%i4) sqrt(Tp(t) . Tp(t))/sqrt(rp(t) . rp(t));
(%o4)
```

$$\frac{\sqrt{\frac{(\sin t)^2}{2} + \frac{(\cos t)^2}{2}}}{\sqrt{(\sin t)^{\,2} + (\cos t)^{\,2} + 1}}$$

```
(%i5) trigsimp(%);
(%o5)
```

$$\frac{1}{2}$$

```
(%i6) define(kappa(t), %);
(%o6)
```

$$\kappa\left(t\right) \;:= \frac{1}{2}$$

We have a lot of practice with derivatives, but we can integrate, too.

```
(%i7) integrate(r(t), t);
```

(%o7)

$$\left[\frac{t^2}{2}, \sin t, -\cos t\right]$$

With integrals we can compute the arc length, but be warned, the arc length may only *rarely* be calculated in closed form. More often than not the arc length can not be represented by an elementary function. We do an example for the sake of argument. We will define a simple vector function, calculate the derivative, and integrate the norm of the derivative.

```
(%i8) g(t) := [2 * t, 3 * sin(t), 3 * cos(t)];
(%o8)
```

$$g(t) := [2t, 3 \sin t, 3 \cos t]$$

```
(%i9) define(gp(t), diff(g(t), t));
(%o9)
```

$$\mathrm{gp}(t) := [2, 3 \cos t, -3 \sin t]$$

```
(%i10) integrate(trigsimp(sqrt(gp(t) . gp(t))), t, 0, 2*%pi);
(%o10)
```

$$2\sqrt{13}\,\pi$$

Note that we used the special notation `%pi` for our favorite mathematical constant. We need the same thing for Euler's constant `%e` and the imaginary unit `%i`.

Also note that we wrapped `sqrt(gp(t) . gp(t))` with `trigsimp` in the `integrate` call. As of the time of this writing, there is a bug in Maxima so that the integral is not computed correctly without the simplification. See Bug ID: 2880797 in the Maxima bug tracker.

Especially with arc lengths sometimes we need to do numerical integration instead of symbolic integration. We can do it with the romberg function.

```
(%i11) romberg(sqrt(gp(t) . gp(t)), t, 0, 2*%pi);
(%o11)
```

$$22.65434679827795$$

3 Functions of Several Variables

Now let's try some multivariate functions.

```
(%i1) f(x,y) := (x^2 - y^2)^2;
```

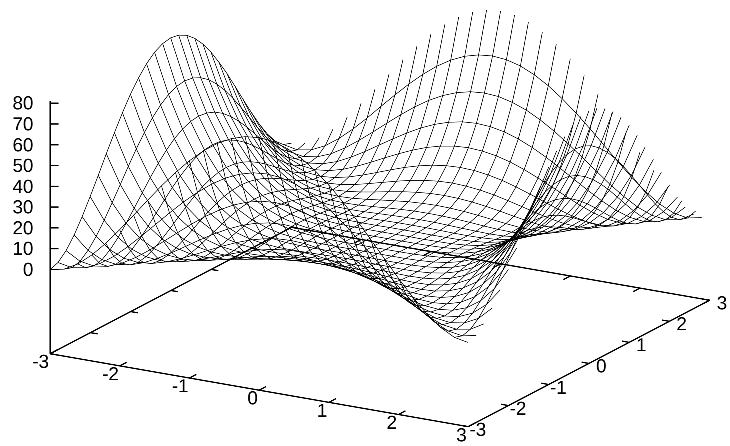

Figure 8: A plain surface plot of a function of two variables

(%o1)

$$f(x,y) := \left(x^2 - y^2\right)^2$$

Let's take a look at a plot.

```
(%i2)  load(draw);
(%o2)
```

/usr/local/share/maxima/5.19.2/share/draw/draw.lisp

```
(%i3)  draw3d(explicit(f(x,y), x, -3, 3, y, -3, 3));
(%o3)
```

[gr3d (explicit)]

See Figure BLANK.

The important part is to enclose the function with `explicit()`. We can get a fancier, colored plot if we put `enhanced3d = true` at the beginning of the function call.

```
(%i4)  draw3d(enhanced3d = true, explicit(f(x,y), x, -3, 3, y, -3, 3));
(%o4)
```

[gr3d (explicit)]

We can see level curves (also known as a contour map) of the function f with the following:

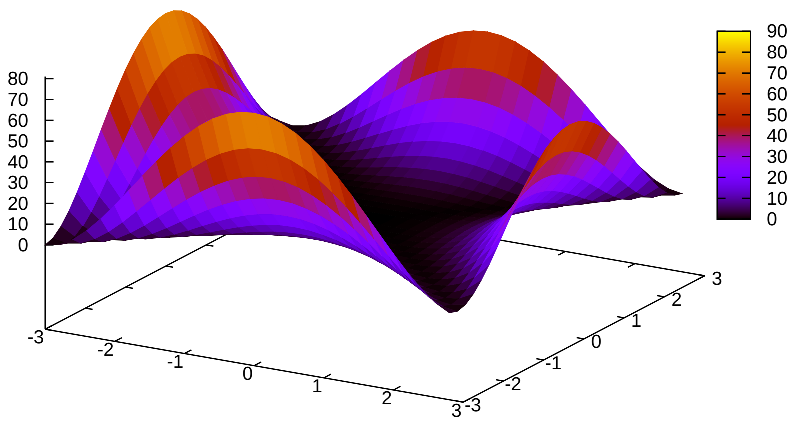

Figure 9: An enhanced surface plot of a function of two variables

```
(%i5)  draw3d(explicit(f(x,y), x, -5, 5, y, -5, 5),
          contour_levels = 15,
          contour        = map);
(%o5)
```

[gr3d (explicit)]

An alternative is to use

```
(%i6)  contour_plot(f(x,y), [x, -5, 5], [y, -5, 5] );
(%o6)
```

false

The contour map is a 2D plot. If we raise the contour lines up to the plot surface then the lines are more precisely called horizontal traces. Here is a plot of these.

```
(%i7)  draw3d(enhanced3d     = true,
          explicit(f(x,y), x, -3, 3, y, -3, 3),
          contour_levels = 15,
          contour        = surface,
          surface_hide   = true);
(%o7)
```

[gr3d (explicit)]

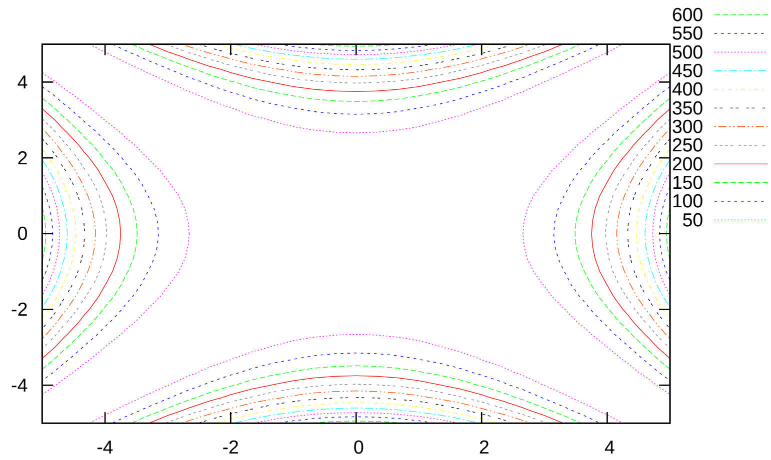

Figure 10: A contour plot, showing level curves of f.

See Figure 6. We do `surfacehide = true` so that we can see the traces better.

3.1 Partial Derivatives

We do partial derivatives in the natural way. For example, for the partial derivative with respect to x we do

```
(%i1)  diff(f(x,y), x);
(%o1)
```

$$\frac{d}{d\,x}\,f\left(x,y\right)$$

The long way to get higher order derivatives is to nest the `diff` calls. The second order partial derivative is

```
(%i2)  diff(diff(f(x,y), x), x);
(%o2)
```

$$\frac{d^2}{d\,x^2}\,f\left(x,y\right)$$

and the second partial with respect to x then y is

```
(%i3)  diff(diff(f(x,y), x), y);
```

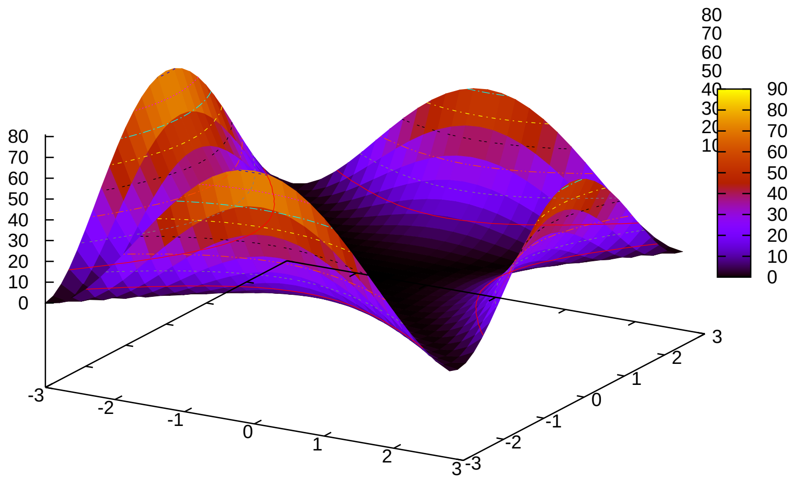

Figure 11: Another form of contour plot which shows horizontal traces on f

(%o3)

$$\frac{d^2}{d\,x\,d\,y}\,f\left(x,y\right)$$

For higher order partial derivatives a quicker way is to do

```
(%i4)  G: x^7 * y^8;
(%o4)
```

$$x^7\,y^8$$

```
(%i5)  diff(G, x, 1, y, 2, x, 3);
(%o5)
```

$$47040\,x^3\,y^6$$

The above first differentiates G with respect to x three times, then with respect to y two times, and finally with respect to x one time; that is, the arguments match the Leibnitz notation for partial derivatives.

3.2 Linear Approximation and Differentials

A function of two variables is differentiable at a point (x_0, y_0) when it is closely approximated by the tangent plane to the curve at (x_0, y_0). We saw in class how to find the tangent plane at (x_0, y_0), and we also discussed how we could use the tangent plane to approximate values of $f(x, y)$ for (x, y) near (x_0, y_0).

We will not bother with the tangent plane in Maxima, but we can quickly find the linear approximation L to f (which is essentially the same thing) by means of the `taylor` function.

Let $f(x, y) = \mathrm{e}^{x^2} \sin(y)$ and let us find the linear approximation of f at the point $(1, 2)$.

```
(%i1) f(x,y) := exp(x^2) * sin(y);
(%o1)
```

$$f(x, y) := \exp x^2 \sin y$$

```
(%i2) taylor(f(x,y), [x,y], [1,2], 1);
(%o2)
```

$$\sin 2\, e + (2 \sin 2\, e\, (x-1) + \cos 2\, e\, (y-2)) + \cdots$$

The function L is shown in the output as everything but the three dots. The ellipsis is a way to indicate that the returned expression is an approximation to the original function.

The linear approximation arguments are all self explanatory except the last. It represents the *order* of the Taylor series. When we find a linear approximation to a differentiable function what we are actually doing is finding a Taylor series of order 1, about the point (x_0, y_0).

We can get the total differential by doing `diff` without specifying any independent variables.

```
(%i3) diff(f(x,y));
(%o3)
```

$$e^{x^2} \cos y \,\mathrm{del}(y) + 2\, x\, e^{x^2} \sin y \,\mathrm{del}(x)$$

The symbols `del(y)` and `del(x)` stand for $\mathrm{d}y$ and $\mathrm{d}x$, respectively.

3.3 Chain Rule and Implicit Differentiation

Suppose we have a function of two variables, $f(x, y) = \mathrm{e}^{x^2} \sin y$.

```
(%i1) f(x,y) := exp(x^2) * sin(y);
(%o1)
```

$$f(x, y) := \exp x^2 \sin y$$

We saw earlier what the partial derivatives of f were with respect to x and y. But suppose x and y are functions of some other variables, for instance,

```
(%i2) [x,y] : [s^2 * t, s * t^2];
```

```
(%o2)
```

$$\left[s^2\,t , s\,t^2\right]$$

Then what are the partial derivatives of f with respect to s and t? Happily for us, Maxima does the chain rule automatically.

```
(%i3) diff(f(x,y), s);
(%o3)
```

$$4\,s^3\,t^2\,e^{s^4\,t^2}\,\sin \left(s\,t^2\right)+t^2\,e^{s^4\,t^2}\,\cos \left(s\,t^2\right)$$

```
(%i4) diff(f(x,y), t);
(%o4)
```

$$2\,s^4\,t\,e^{s^4\,t^2}\,\sin \left(s\,t^2\right)+2\,s\,t\,e^{s^4\,t^2}\,\cos \left(s\,t^2\right)$$

That was easy. But be warned that the derivative with respect to x does not work anymore.

```
(%i5) diff(f(x,y), x);
```

** error while printing error message **
~:M: second argument must be a variable; found ~M
– an error. To debug this try debugmode(true);

We could fix this by using different letters, `u` and `v`:

```
(%i6) diff(f(u,v), u);
(%o6)
```

$$2\,u\,e^{u^2}\,\sin v$$

But that is cheating. A better way to fix it is to kill the relationship between `x` and `s`, `t`.

```
(%i7) kill(x, y);
(%o7)
```

done

```
(%i8) diff(f(x,y), x);
(%o8)
```

$$2\,x\,e^{x^2}\,\sin y$$

Implicit differentiation is relatively easy, given what we have already done. For example, let's find the first partial derivatives of z when $xyz + x^2y^3z^4 = x + xz$.

```
(%i9)  F: x*y*z + x^2*y^3*z^4 - x - x*z;
(%o9)
```

$$x^2\, y^3\, z^4 + x\, y\, z - x\, z - x$$

```
(%i10)  Fx: diff(F, x);
(%o10)
```

$$2\, x\, y^3\, z^4 + y\, z - z - 1$$

```
(%i11)  Fy: diff(F, y);
(%o11)
```

$$3\, x^2\, y^2\, z^4 + x\, z$$

```
(%i12)  Fz: diff(F, z);
(%o12)
```

$$4\, x^2\, y^3\, z^3 + x\, y - x$$

```
(%i13)  [-Fx/Fy, -Fy/Fz];
(%o13)
```

$$\left[\frac{-2\, x\, y^3\, z^4 - y\, z + z + 1}{3\, x^2\, y^2\, z^4 + x\, z}, \frac{-3\, x^2\, y^2\, z^4 - x\, z}{4\, x^2\, y^3\, z^3 + x\, y - x}\right]$$

3.4 Directional Derivatives and the Gradient

Remember to do `load(vect)` if you haven't already during the session. By default Maxima assumes that the coordinate system is rectangular (cartesian) in the variables x, y, and z. If you do not want that then you need to change it by means of the `scalefactors` command.

Given the function of two variables, $f(x, y) = e^{x^2} \sin y$.

```
(%i1)  f(x,y) := exp(x^2) * sin(y);
(%o1)
```

$$f(x, y) := \exp x^2\, \sin y$$

We change to the correct coordinate system.

```
(%i2)  load(vect);
```

(%o2)

/usr/local/share/maxima/5.19.2/share/vector/vect.mac

```
(%i3) scalefactors([x,y]);
```

(%o3)

done

Next we find the gradient.

```
(%i4) gdf: grad(f(x,y));
```

(%o4)

$$\text{grad}\left(e^{x^2}\,\sin y\right)$$

```
(%i5) ev(express(gdf), diff);
```

(%o5)

$$\left[2\,x\,e^{x^2}\,\sin y, e^{x^2}\,\cos y\right]$$

```
(%i6) define(gdf(x,y), %);
```

(%o6)

$$\text{gdf}\,(x,y)\ :=\left[2\,x\,e^{x^2}\,\sin y, e^{x^2}\,\cos y\right]$$

The directional derivative of f at the point $(1,2)$ in the direction of the vector $\mathbf{v} = \langle 3,4 \rangle$ is

```
(%i7) v: [3,4];
```

(%o7)

$$[3,4]$$

```
(%i8) (gdf(1,2) . v)/sqrt(v . v);
```

(%o8)

$$\frac{6\,e\,\sin 2+4\,e\,\cos 2}{5}$$

```
(%i9) ev(%, diff);
```

(%o9)

$$\frac{6\,e\,\sin 2+4\,e\,\cos 2}{5}$$

```
(%i10) float(%);
```

(%o10)

$$2.061108499400332$$

We know from our theory that the directional derivative is maximized when **v** points in the direction of the gradient, and the maximum value is the length of the gradient vector. Let's see just how big that is.

```
(%i11) sqrt(gdf(1,2) . gdf(1,2));
(%o11)
```

$$\sqrt{4\,e^2\,(\sin 2)^2 + e^2\,(\cos 2)^2}$$

```
(%i12) float(ev(%, diff));
(%o12)
```

$$5.071228088168654$$

3.5 Optimization and Local Extrema

To find critical points we need to solve the system of equations $f_x = 0$ and $f_y = 0$. Let $f(x,y) = 2x^4 + 2y^4 - 8xy$.

```
(%i1) f(x,y) := 2 * x^4 + 2 * y^4 - 8 * x * y;
(%o1)
```

$$f(x,y) := 2\,x^4 + 2\,y^4 + (-8)\,x\,y$$

Let's take a look at a plot of f.

```
(%i2) load(draw);
(%o2)
```

`/usr/local/share/maxima/5.19.2/share/draw/draw.lisp`

```
(%i3) draw3d(enhanced3d = true, explicit(f(x,y), x, -2, 2, y, -2, 2));
(%o3)
```

[gr3d (explicit)]

See Figure BLANK.

Now let's take a look at a contour plot of f. We do it just like the plot of the surface, except we use the argument `contour = map`.

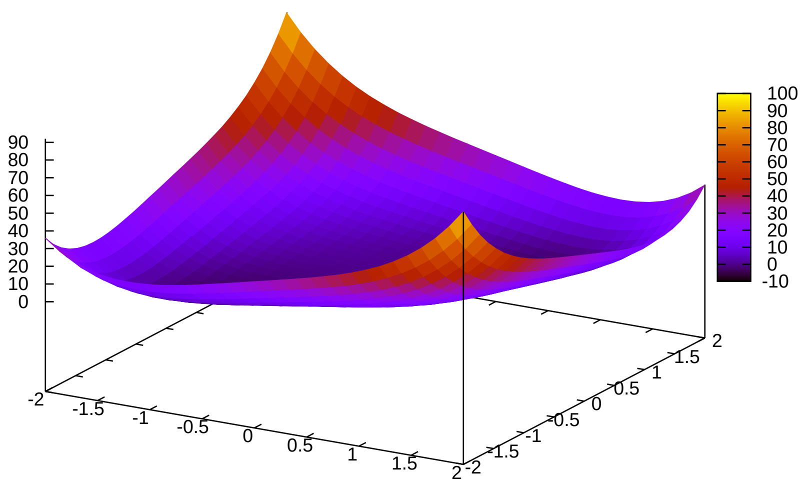

Figure 12: A surface plot of f

```
(%i4)  draw3d(explicit(f(x,y), x, -2, 2, y, -2, 2),
          contour        = map);
(%o4)
```

$$[gr3d\,(explicit)]$$

See Figure BLANK.

Now let's find the first order partial derivatives of f, set them equal to zero, and solve for values of (x, y). We do this with the `solve` function, which assumes that the expressions are set equal to zero by default. Keep in mind that `solve` finds all real and complex solutions; we only care about the real valued solutions, however, and will ignore the rest.

```
(%i5)  fx : diff(f(x,y), x);
(%o5)
```

$$8\,x^3 - 8\,y$$

```
(%i6)  fy : diff(f(x,y), y);
(%o6)
```

$$8\,y^3 - 8\,x$$

```
(%i7)  solve([fx,fy], [x,y]);
```

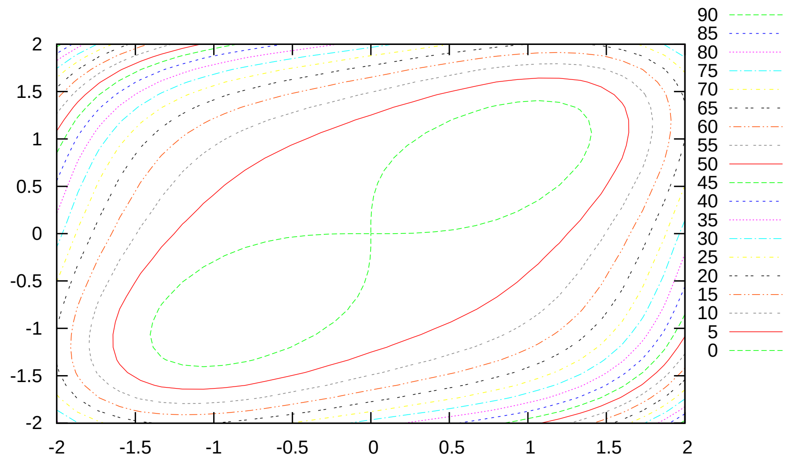

Figure 13: The level curves of f which suggest two extrema and a saddle point

(%o7)

$$\left[\left[x=(-1)^{\frac{1}{4}}\,i, y=-(-1)^{\frac{3}{4}}\,i\right], \left[x=-(-1)^{\frac{1}{4}}, y=-(-1)^{\frac{3}{4}}\right], \left[x=-(-1)^{\frac{1}{4}}\,i, y=(-1)^{\frac{3}{4}}\,i\right], \left[x=(-1)^{\frac{1}{4}}, y=(-1)^{\frac{3}{4}}\right], [x=-i, y=i], [x=i, y=-i], [x=-1, y=-1], [x=1, y=1], [x=0, y=0]\right]$$

Critical points are at $(0,0)$, $(1,1)$, and $(-1,-1)$. (Note that both partial derivatives exist everywhere.) We need to see what the Hessian says at those locations.

```
(%i8)  H: hessian(f(x,y), [x,y]);
(%o8)
```

$$\begin{pmatrix} 24\,x^2 & -8 \\ -8 & 24\,y^2 \end{pmatrix}$$

```
(%i9)  determinant(H);
(%o9)
```

$$576\,x^2\,y^2-64$$

We do the Second Derivative Test by plugging in the above three points into f_{xx} and det(H). (Of course we can do it mentally but let's try it with Maxima.) A quick way to plug numbers into expressions is with the `subst` function.

```
(%i10) subst([x = -1, y = -1], diff(fx, x));
(%o10)
```

$$24$$

```
(%i11) subst([x = -1, y = -1], determinant(H));
(%o11)
```

$$512$$

From the above we conclude that $(-1, -1)$ is a local minimum of f, and the value is

```
(%i12) f(-1,-1);
(%o12)
```

$$-4$$

Doing the same for $(0, 0)$ and $(1, 1)$ shows that $(0, 0)$ is a saddle point and $(1, 1)$ is also a local minimum (we know this already by symmetry).

3.6 Lagrange Multipliers

Find the extreme values of the function $f(x, y) = 2x^2 + y^2$ on the circle $x^2 + y^2 = 1$.

```
(%i1) f(x,y) := 2 * x^2 + y^2;
(%o1)
```

$$f(x, y) := 2\,x^2 + y^2$$

```
(%i2) g: x^2 + y^2;
(%o2)
```

$$y^2 + x^2$$

We set up the system of equations `grad(f) = h * grad(g), g = 1`. (We do not use "`lambda`" because that name is already reserved for an existing function in Maxima.)

```
(%i3) eq1: diff(f(x,y), x) = h * diff(g, x);
(%o3)
```

$$4\,x = 2\,h\,x$$

```
(%i4) eq2: diff(f(x,y), y) = h * diff(g, y);
(%o4)
```

$$2\,y = 2\,h\,y$$

```
(%i5) eq3: g = 1;
```

(%o5)

$$y^2 + x^2 = 1$$

Now we solve the system for x, y, and h.

```
(%i6) solve([eq1, eq2, eq3], [x, y, h]);
(%o6)
```

$$[[x = 1, y = 0, h = 2], [x = -1, y = 0, h = 2], [x = 0, y = -1, h = 1], [x = 0, y = 1, h = 1]]$$

We see that the extreme values lie among $(1, 0)$, $(-1, 0)$, $(0, -1)$, $(0, 1)$.

```
(%i7) [f(1,0), f(-1,0), f(0,-1), f(0,1)];
(%o7)
```

$$[2, 2, 1, 1]$$

So the minima occur at $(1, 0)$ and $(-1, 0)$; the maxima occur at $(0, -1)$ and $(0, 1)$.

4 Multiple Integration

4.1 Double Integrals

We can iterate `integrate` calls. For example, suppose we wanted to calculate

$$\int \int \left(x^3 - 3xy\right) \, \mathrm{d}y \, \mathrm{d}x.$$

```
(%i1) f(x,y) := x^3 - 3*x*y;
(%o1)
```

$$f(x, y) := x^3 - 3\, x\, y$$

```
(%i2) integrate(integrate(f(x,y), y), x);
(%o2)
```

$$\frac{x^4\, y}{4} - \frac{3\, x^2\, y^2}{4}$$

Maxima does not provide arbitrary constants of integration; the user must remember them. It is easy to do definite integration, for example, we could do

$$\int_0^1 \int_{\sqrt{x}}^{2-x} \left(x^3 - 3xy\right) \, \mathrm{d}y \, \mathrm{d}x.$$

with

```
(%i3) integrate(integrate(f(x,y), y, x^1/2, 2 - x), x, 0, 1);
(%o3)
```

$$-\frac{173}{160}$$

4.2 Integration in Polar Coordinates

We can integrate in polar coordinates in the obvious way. We simply make the substitution `x = rcos(theta)` and `y = rsin(theta)`, then don't forget to multiply the integrand by `r`.

```
(%i1) f(x,y) := x^2 + y^2;
(%o1)
```

$$f(x,y) := x^2 + y^2$$

```
(%i2) [x,y]: [r * cos(theta), r * sin(theta)];
(%o2)
```

$$[r\cos\vartheta, r\sin\vartheta]$$

```
(%i3) integrate(integrate(f(x,y) * r, r, 0, 2*cos(theta)), theta, -%pi/2,
%pi/2);
(%o3)
```

$$\frac{3\pi}{2}$$

4.3 Triple Integrals

Triple integrals are done just like double integrals: by nested `integrate` calls.

Let's do

$$\int_0^1 \int_0^{-x} \int_0^{x+y} x^2 yz \, dz \, dy \, dx.$$

```
(%i1) integrate(integrate(integrate(x^2*y*z,z,0,x+y),y,0,-x),x,0,1);
(%o1)
```

$$\frac{1}{168}$$

4.4 Integrals in Cylindrical and Spherical Coordinates

These integrals are computed just like ordinary triple integrals except we multiply the integrand by r (in cylindrical coordinates) or $\rho^2 \sin(\phi)$ (in spherical coordinates). See the next section for a more general way of doing this. It is sometimes useful to use plots to decide how to represent the region of integration.

Let's do an integral in cylindrical coordinates.

$$\int_{-2}^{2} \int_{0}^{\sqrt{4-x^2}} \int_{0}^{3} yz \, \mathrm{d}z \, \mathrm{d}y \, \mathrm{d}x.$$

Note that r goes from 0 to 2 and θ goes from 0 to π.

```
(%i1) f(x,y,z) := y*z;
(%o1)
```

$$f(x, y, z) := y\, z$$

```
(%i2) [x,y,z] : [r*cos(theta), r*sin(theta), z];
(%o2)
```

$$[r\, \cos\vartheta, r\, \sin\vartheta, z]$$

```
(%i3) integrate(integrate(integrate(f(x,y,z)*r, z,0,3), r,0,2), theta,0,%pi);
(%o3)
```

$$24$$

Now let's do an integral in spherical coordinates.

$$\int_{-1}^{1} \int_{0}^{\sqrt{1-x^2}} \int_{0}^{\sqrt{1-x^2-y^2}} xz \, \mathrm{d}z \, \mathrm{d}y \, \mathrm{d}x.$$

Note that ρ goes from 0 to 1, θ goes from 0 to π, and ϕ goes from 0 to $\pi/2$.

```
(%i4) kill(f,x,y,z);
(%o4)
```

done

```
(%i5) f(x,y,z) := x*z;
(%o5)
```

$$f(x, y, z) := x\, z$$

```
(%i6) [x,y,z] : [rho*sin(phi)*cos(theta), rho*sin(phi)*sin(theta), rho*cos(theta)];
(%o6)
```

$$[\sin\varphi\, \rho\, \cos\vartheta, \sin\varphi\, \rho\, \sin\vartheta, \rho\, \cos\vartheta]$$

```
(%i7) integrate(integrate(integrate(f(x,y,z)*rho^2*sin(phi),rho,0,1),theta,0,%pi),
phi,0,%pi/2);
(%o7)
```

$$\frac{\pi^2}{40}$$

4.5 Change of Variables

For more general transformations $x = x(u, v)$ and $y = y(u, v)$ we can use the `jacobian` function in the `linearalgebra` package, which is loaded by default. Let $f(x, y) = x + y$, and we will calculate

$$\int\int_R (x + y)\,\mathrm{d}A.$$

```
(%i1) f(x,y) := x + y;
(%o1)
```

$$f(x, y) := x + y$$

Let's make the transformation $x = u^3 - v^4$ and $y = 5uv$.

```
(%i2) [x,y]: [u^3 - v^4, 5 * u * v];
(%o2)
```

$$\left[u^3 - v^4, 5\,u\,v\right]$$

We need the Jacobian:

```
(%i3) J: jacobian([x,y], [u,v]);
(%o3)
```

$$\begin{pmatrix} 3\,u^2 & -4\,v^3 \\ 5\,v & 5\,u \end{pmatrix}$$

```
(%i4) J: determinant(J);
(%o4)
```

$$20\,v^4 + 15\,u^3$$

We were lucky in this example because the Jacobian is positive as long as u is positive (or just not terribly negative). If this had not happened then we would need to be careful about the sign of J, which in principle could be a problem because Maxima does not integrate absolute value. Nevertheless, we finish up with the integration. It takes the form (we will make up random limits of integration, but in a given problem we would need to determine these)

$$\int_3^4 \int_1^2 f\left(x(u,v), y(u,v)\right) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \, \mathrm{d}u \, \mathrm{d}v.$$

```
(%i5) integrate(integrate(f(x,y) * J, u,1,2), v,3,4);
(%o5)
```

$$-\frac{113349305}{252}$$

5 Vector Calculus

5.1 Vector Fields

A *vector field* is a vector function defined on a subset of $\mathbb{R}^3$ (or $\mathbb{R}^2$) that assigns a vector (of the same dimension) with each point in the set.

Two-dimensional

A two dimensional vector field associates a two-dimensional vector with each point in the plane (or a subset thereof). We may use the `draw` package to plot vector fields with Maxima.

```
load(draw);
```

Let us plot the 2D vector field $\mathbf{F}(x,y) = \langle \cos y, x \rangle$.

The important parts of the following code are the first line where we specify that plotting should be done from `-6` to `6` and in the `vf2d` line where we specify function values `[cos(y),x]` in the second list. The division by `6` in that same line is just to make the arrows smaller so that they may more easily be seen. The rest of the code is boilerplate and may be copied/pasted verbatim.

```
coord: setify(makelist(k,k,-4,4));
points2d: listify(cartesian_product(coord,coord));
vf2d(x,y):= vector([x,y],[cos(y),x]/6);
vect2: makelist(vf2d(k[1],k[2]),k,points2d);
apply(draw2d, append([color=blue], vect2));
```

See Figure BLANK.

Gradient vector fields

The gradient vector ∇f defines a vector field on the domain of f. We can plot this vector field in two ways: with the `draw` package, or with the `ploteq` function.

First we need a gradient function. Let's start with $f(x,y) = x^2 - y^2$. Then, of course, $\nabla f = \langle 2x, \ -2y \rangle$.

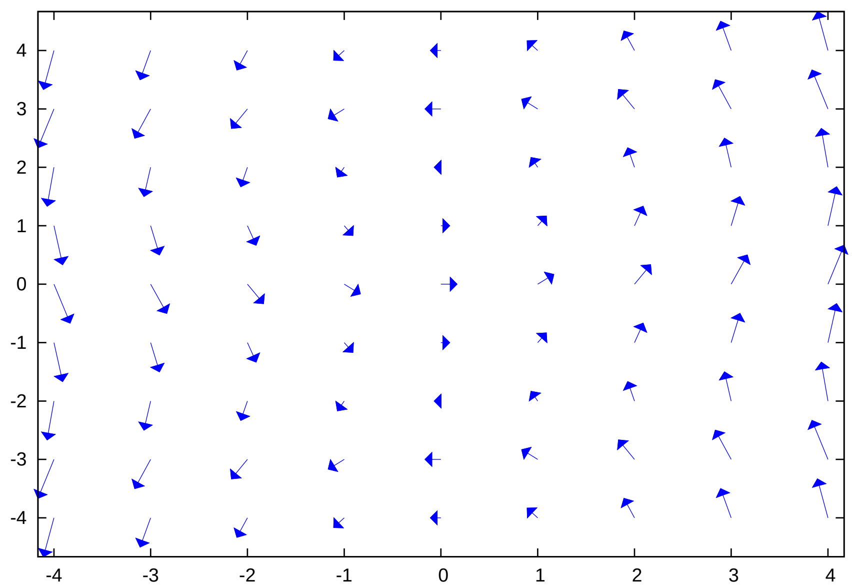

Figure 14: A plot of the vector field $\mathbf{F}(x, y) = \langle \cos y, x \rangle$

```
kill(f, x, y, gdf);
f(x,y) := x^2 - y^2;
scalefactors([x,y]);
gdf(x,y) := grad(f(x,y));
ev(express(gdf(x,y)), diff);
define(gdf(x,y), %);
```

Next we make the plot. (We have omitted the first few lines because we will use the same values from the last example. However, these lines should not ordinarily be skipped).

```
vf2d(x,y) := vector([x,y], gdf(x,y)/8);
vect2: makelist(vf2d(k[1],k[2]),k,points2d);
apply(draw2d, append([head_length=0.1, color=blue], vect2));
```

Alternatively, we can do it in one line with the `ploteq` function as long as we remember to put a minus sign in front of the original function (`ploteq` plots vectors that are the opposite of gradient vectors).

```
ploteq(-(x^2+y^2),[x,y],[x,-4,4],[y,-4,4],[vectors,"blue"]);
```

Three-dimensional

This is almost identical to the 2D case, except that we need to account for z in the code and use `draw3d` instead of `draw2d`.

Note that with 3d plots the graph can be dynamically moved around with the mouse to look from different angles. Also note that the arrows in a vector field plot are often times difficult to see because of their length. I usually scale them down to see the relationships better.

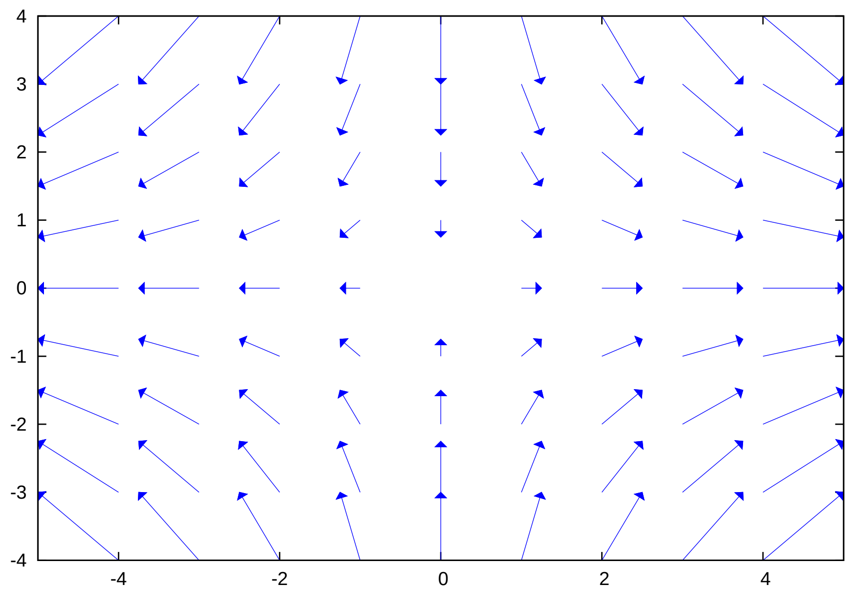

Figure 15: The gradient vector field ∇f for the function $f(x, y) = x^2 - y^2$

Let's plot the vector field

$$\mathbf{F}(x, y, z) \ = \langle z, x, y \rangle.$$

```
coord: setify(makelist(k,k,-2,2));
points3d: listify(cartesian_product(coord,coord,coord));
vf3d(x,y,z):= vector([x,y,z],[z,x,y]/8);
vect3: makelist(vf3d(k[1],k[2],k[3]),k,points3d);
apply(draw3d, append([color=blue], vect3));
```

See Figure BLANK.

5.2 Line Integrals

We are given a smooth space curve C defined by the parametric equations $x = x(t)$, $y = y(t)$, $z = z(t)$, for $a \leq t \leq b$, and a function f which is defined on C. Sometimes we work with a vector function $\mathbf{r}$ instead of the parametric equations:

$$\mathbf{r}(t) \ = \langle x(t),\ y(t),\ z(t) \rangle\,, \quad \text{for } a \leq t \leq b.$$

With respect to arc length

These integrals are of the form

$$\int_C f(x, y, z)\, \mathrm{d}s$$

which can be reparameterized as

$$\int_a^b f\left(x(t), y(t), z(t)\right) \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2}\, \mathrm{d}t$$

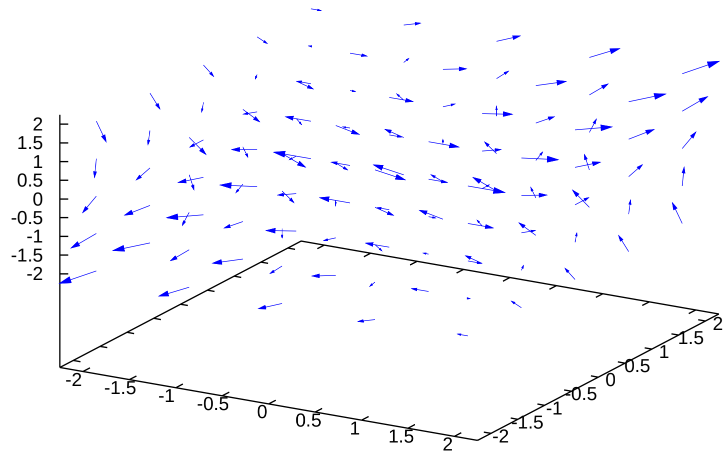

Figure 16: A plot of the vector field $\mathbf{F}(x, y, z) = \langle z, x, y\rangle$

or more compactly as

$$\int_a^b f(\mathbf{r}(t))\,|\mathbf{r}'(t)|\,dt.$$

For example, let's start with $f(x, y) = x^2 + y^2$. We will integrate along the curve C parameterized by $\langle \cos t,\ \sin 2t\rangle$ for $0 \le t \le 1$.

```
(%i1) f(x,y) := x^2 + y^2;
(%o1)
```

$$f(x, y) := x^2 + y^2$$

```
(%i2) [x,y]: [cos(t), sin(2*t)];
(%o2)
```

$$[\cos t, \sin(2\,t)]$$

```
(%i3) rp: diff([x,y], t);
(%o3)
```

$$[-\sin t, 2\,\cos(2\,t)]$$

```
(%i4) romberg(f(x,y)*sqrt(rp . rp), t, 0, 1);
(%o4)
```

$$1.635879048260743$$

Of vector fields

Here we are trying to compute integrals of the form

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_C \mathbf{F}\left(\mathbf{r}(t)\right) \cdot \mathbf{r}'(t)\,\mathrm{d}t,$$

alternately written

$$\int_C P\,\mathrm{d}x + Q\,\mathrm{d}y + R\,\mathrm{d}z, \text{ where } \mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}.$$

Let's work with the vector field $\mathbf{F}(x, y, z) = \langle -xy^3,\ xz,\ yz^2\rangle$. We will integrate along the curve C parameterized by $\langle t^2,\ t^3,\ t^4\rangle$ for $0 \le t \le 1$.

```
(%i5) F(x,y,z) := [-x*y^3, x*z, y*z^2];
(%o5)
```

$$F(x, y, z) := \left[(-x)\,y^3, x\,z, y\,z^2\right]$$

```
(%i6) [x,y,z]: [t^2, t^3, t^4];
(%o6)
```

$$\left[t^2, t^3, t^4\right]$$

```
(%i7) romberg(F(x,y,z) . diff([x,y,z], t), t, 0, 1);
(%o7)
```

$$.4461538461603605$$

5.3 Conservative Vector Fields and Finding Scalar Potentials

We know from theory that a vector field $\mathbf{F}$ is *conservative* if there exists a function f such that $\mathbf{F} = \nabla f$. Further, we know that fields defined on suitably nice regions are conservative if they are irrotational.

We can check whether a field is conservative with the `curl` function in the `vect` package. For example, let's check the field $\mathbf{F}(x, y) = (4x^3 - 5y^2)\mathbf{i} + (5y^3 - 3x)\mathbf{j}$.

```
(%i1) F(x,y) := [4*x^3 - 5*y^2, 5*y^3 - 3*x];
(%o1)
```

$$F(x, y) := \left[4\,x^3 - 5\,y^2, 5\,y^3 - 3\,x\right]$$

```
(%i2) load(vect);
(%o2)
```

```
/usr/local/share/maxima/5.19.2/share/vector/vect.mac
```

```
(%i3) scalefactors([x,y]);
```

(%o3)

$$\mathbf{done}$$

```
(%i4) curl(F(x,y));
```

(%o4)

$$\mathrm{curl}\left(\left[4\,x^3-5\,y^2,5\,y^3-3\,x\right]\right)$$

```
(%i5) express(%);
```

(%o5)

$$\frac{d}{d\,x}\,\left(5\,y^3-3\,x\right)-\frac{d}{d\,y}\,\left(4\,x^3-5\,y^2\right)$$

```
(%i6) ev(%, diff);
```

(%o6)

$$10\,y-3$$

Since the `curl` is not zero, the field is not conservative. How about

$$\mathbf{F}(x,y) \;= (x^3+5y)\mathbf{i}+(5y^3+5x)\mathbf{j}?$$

```
(%i7) F(x,y) := [x^3 + 5*y, 5*y^3 + 5*x];
```

(%o7)

$$F\,(x,y)\;:=\left[x^3+5\,y,5\,y^3+5\,x\right]$$

```
(%i8) ev(express(curl(F(x,y))), diff);
```

(%o8)

$$0$$

Since the `curl` is zero, this field is conservative. So the function f satisfying $\mathbf{F} = \nabla f$ exists. We can find the scalar potential f in Maxima with the `potential` function (also in the `vect` package).

Note, however, that because of a bug in Maxima at the time of this writing we need to do a little fancy footwork. We cannot use the letter `x` in the function call; instead we will change it to another letter. When we do that, we must follow it by a call to `scalefactors`.

```
(%i9) F(u,v) := [u^3 + 5*v, 5*v^3 + 5*u];
```

(%o9)

$$F\,(u,v)\;:=\left[u^3+5\,v,5\,v^3+5\,u\right]$$

```
(%i10) scalefactors([u,v]);
```

(%o10)

$$\mathbf{done}$$

```
(%i11) potential(F(u,v));
```

(%o11)

$$\frac{5\,v^4+20\,u\,v+u^4}{4}$$

We can easily check that $f(x,y) = (5x^4+20xy+y^4)/4$ satisfies $\mathbf{F} = \nabla f$. The fundamental theorem for line integrals now allows us to compute line integrals that look like

$$\int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} \;=\; \int_C \nabla f\cdot \mathrm{d}\mathbf{r},$$

which simplifies to $f(\mathbf{r}(b)) - f(\mathbf{r}(a))$. So, for instance, for a curve C with initial point $(0,1)$ and terminal point $(2,3)$ we have

$$\int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} \;=\; \int_C \left[(x^3+5y)\mathbf{i} + (5y^3+5x)\mathbf{j}\right]\cdot \mathrm{d}\mathbf{r}$$

to be equal to (using the result from `potential`, above)

```
(%i12) define(f(u,v), %);
```

(%o12)

$$f\left(u,v\right) := \frac{5\,v^4+20\,u\,v+u^4}{4}$$

```
(%i13) f(2,3) - f(0,1);
```

(%o13)

$$134$$

All of the above can be done in three dimensions, too. We need only to do `scalefactors([x,y,z])` and `scalefactors([u,v,w])`, when appropriate.

6 Miscellaneous

6.1 Saving your plots

In most Microsoft Windows® applications, you can save a plot either with a direct option in a File menu, or if nothing else, by right-clicking on the plot and selecting "Copy". Unfortunately, with the plots described in this document (and on Linux, at least), this is not possible. Instead, the user must separately send the plot to a file.

In the `draw` package plots can be saved with the `terminal` argument. The following are a couple of examples of what this looks like, and see the documentation for `terminal` for more details and options. For a standard 2d plot you can usually get by with something like this:

```
draw3d(enhanced3d = true,
   implicit(plane1, x,-4,4, y,-4,4, z, -6,6),
   terminal=eps_color, file_name="plot");
```

But for the more complicated vector field plots you need something like this:

```
apply(draw3d,
      append([color=blue,
              terminal=eps_color,
              file_name="plot"],
      vect3));
```

6.2 Saving your Maxima commands

Once you type a bunch of commands into Maxima you will often want to save those commands into a file to be opened and used again later. The following command will pick out all of the Maxima commands from the session and save them into a text file that can be opened later with any text editor.

Note that only commands since the last `kill(all)` statement will be saved.

```
stringout("nameoffile.txt", input, file_output_append = true);
```

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